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Remarks about the Construction of Optimal Subspaces of Approximants of a Hilbert Space

JEAN-PIERRE AUBIN

Division of Mathematical Sciences, Purdue University, Lafayette, Indiana 47907

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INTRODUCTION

We study in this paper the problem of the construction of subspaces of approximants of a Hilbert space V defined as the domain of linear operators.

Usually, we introduce "a priori" subspaces of approximants and we study their properties. For instance, if the space V is a space of functions or of distributions, we choose spaces of approximants which are polynomials or piecewise-polynomials (spline functions).

Another way of attacking this problem is to construct subspaces of approximants which satisfy a given set of properties.

Consider, e.g., the so-called problem of "optimal interpolation" (see [1] and its references). We have seen that if V is a Sobolev space $H^m(R^n)$, the solutions of this problem are piecewise-polynomials of degree $2m - 1$ if $n = 1$, but are linear combinations of the translations of the elementary solution of $(-\Delta + \lambda)^m$ if $n > 1$ (Δ denotes the Laplacian).

Below, we shall study a more general problem which is better adapted to the needs of the theory of approximation of solutions of linear problems.

The space V (of functions) we use is a Hilbert space, the domain of (one or) several operator A^i mapping V into a space F^i.

The data of the problem are the following:

We introduce "discrete analogues" of the above items: a space V_h (of sequences) and operators A_h^i mapping V_h into space F_h^i.

Moreover, we assume that there exist operators r_h^i which associate with $f^i \in F^i$ a discrete element $f_h^i = r_h^i f^i \in F_h^i$.

Finally, we supply V with a positive Hermitian bilinear form $((u, v))$ and its associated seminorm $\| u \| = ((u, u))^{1/2}$.

The question we ask is:

Characterize the subspace of approximants u of V (if any) satisfying

(i) $r_h{}^i A^i u = A_h{}^i u_h$ (for all i),

(ii) $\| u \| \leqslant \| v \|$ for every v such that $r_h{}^i A^i v = A_h{}^i u_h$ for all i, where u_h ranges over V_h.

The problem of "optimal interpolation" is the particular case where $F^i = V$, $F_h{}^i = V_h$, $r_h{}^i = r_h$ (for every i) and where the operators A^i and $A_h{}^i$ are the identity mappings.

We shall give several characterizations of the solutions of this problem and deduce several sufficient conditions for existence and uniqueness. In particular, we shall prove "commutation" formulas which are useful for proving convergence theorems.

Among the examples we list below, we find subspaces of approximants we have already used for approximating solutions of differential problem by finite-differences schemes (cf. [2, 4]).

1. General situation

Let V and F be two Hilbert spaces and A a linear operator from V into F. Let us associate with a parameter h discrete spaces V_h and F_h and a linear operator from V_h into F_h.

We introduce a linear operator $r_h{}^0$ from F into F_h, and a continuous positive Hermitian bilinear form $((u, v))$. We denote by $\| u \| = ((u, u))^{1/2}$ the associated seminorm.

Remark 1.1. This situation contains the case where V (resp. V_h) is the domain of several operators A^i (resp. $A_h{}^i$) mapping V (resp. V_h) into F^i (resp. $F_h{}^i$). We then take $F = \prod F^i$, $F_h = \prod F_h{}^i$, $A = XA^i$ and $A_h = XA_h{}^i$. (We denote by XA^i the operator defined by $XA^i(u) = (A^i u)_i \in \prod F^i$.) ∎

Let us denote by V' the dual of V, by (f, v) the duality pairing on $V' \times V$ and by J the continuous linear operator from V into V' defined by

$$(Ju, v) = ((u, v)) \qquad \text{for all} \quad u, v \in V. \tag{1-1}$$

Our problem is: Characterize the subset $p_h u_h$ of V defined by $u \in p_h u_h$ if and only if

$$\begin{aligned} &\text{(i)} \qquad r_h{}^0 A u = A_h u_h, \\ &\text{(ii)} \qquad \| u \| \leqslant \| v \| \text{ for every } v \text{ such that } r_h{}^0 A v = A_h u_h. \end{aligned} \tag{1-2}$$

We shall deduce our results from the following theorem:

THEOREM 1.1. *Let us assume that the range of* $r_h{}^0A$ *is closed.*

An element u *of* V *belongs to* $p_h u_h$ *if and only if there exists an* $f_h \in F_h'$ *such that*

$$\begin{aligned} &\text{(i)} & Ju &= A' r_h^{0\prime} f_h\,, \\ &\text{(ii)} & r_h{}^0 Au &= A_h u_h\,. \end{aligned} \tag{1-3}$$

Proof. If u is a solution of the system (1-3), u satisfies (1-2)(i). On the other hand, if $r_h{}^0Av = 0$, we obtain

$$\begin{aligned} \| u \|^2 &= (Ju, u) = (A' r_h^{0\prime} f_h\,, u) = (f_h\,, r_h{}^0 A(u + v)) \\ &= (Ju, u + v) \leqslant \| u \| \, \| u + v \|. \end{aligned} \tag{1-4}$$

Since any solution of Eq. (1-2)(i) is equal to $u + v$ where $r_h{}^0Av = 0$, we have obtained Eq. (1-2)(ii).

Conversely, let us assume that $u \in p_h u_h$. Then if $v \in \ker(r_h{}^0A)$, we deduce from Eqs. (1-2) that

$$\lambda^{-1}(\| u \|^2 - \| u + \lambda v \|^2) \leqslant 0 \qquad \text{for any} \quad v \in \ker(r_h{}^0A). \tag{1-5}$$

Letting λ converge to 0, we deduce that

$$((u, v)) = (Ju, v) = 0 \qquad \text{for every} \quad v \in \ker(r_h{}^0A). \tag{1-6}$$

In other words, Ju belongs to the annihilator of $\ker(r_h{}^0A)$ which is equal to the range of its transpose $A' r_h^{0\prime}$, since the range of $r_h{}^0A$ (and thus, the range of $A' r_h^{0\prime}$) is closed.

Therefore, there exists a solution f_h of Eq. (1-3)(i).

COROLLARY 1.1. *Let* N *be the sbuspace* $\{u \in V : \| u \| = 0\}$. If

$$N \cap \ker(r_h{}^0A) = 0,$$

there exists at most one solution of Eq. (1-2).

Proof. If u and v belong to $p_h u_h$, then $u - v$ belongs to $\ker(r_h{}^0A)$ and $J(u - v)$ belongs to the annihilator of $\ker(r_h{}^0A)$. Therefore

$$\| u - v \|^2 = (J(u - v), u - v) = 0 \qquad \text{and} \qquad u - v \in N \cap \ker(r_h{}^0A) = 0.$$

COROLLARY 1.2. *Let us assume that the range* $G(A_h)$ *of* A_h *is contained in the closed range* $G(r_h{}^0A)$ *of* $r_h{}^0A$. *Assume, also, that* $N \cap \ker(r_h{}^0A) = 0$ *and that* $\ker(r_h{}^0A)$ *is complete for the norm* $\| u \|$.

Then $p_h u_h$ contains a unique element u and p_h is a linear operator from V_h into V such that

$$\begin{aligned} &\text{(i)} \quad r_h{}^0 A p_h u_h = A_h u_h\,, \\ &\text{(ii)} \quad \| p_h u_h \| \leqslant \| v \| \text{ for every } v \text{ such that } r_h{}^0 A v = A_h u_h\,. \end{aligned} \tag{1-7}$$

Proof. The first assumption implies that there exists at least one solution of the Eq. (1-2)(i). Let w be such a solution. On the other hand since $\operatorname{Ker}(r_h{}^0 A)$ is complete, there exists a unique orthogonal projection v of w onto $\ker(r_h{}^0 A)$. Then:

$$((w - v, z)) = (J(w - v), z) = 0 \qquad \text{for every} \quad z \in \ker(r_h{}^0 A). \tag{1-8}$$

Therefore $u = w - v$ satisfies Eq. (1-2)(i) and $Ju = J(w - v)$ belongs to the range of $A' r_h^{0\prime}$ (equal to the annihilator of $\ker(r_h{}^0 A)$). Then u belongs to $p_h u_h$ by Theorem 1.1 and is unique by Corollary 1.1.

Finally, we obtain the following corollary:

COROLLARY 1.3. *Let us assume that* $\| u \|$ *is a norm* (i.e., $N = 0$) *and that* $r_h{}^0 A$ *maps* V *onto the Hilbert space* F_h. *Then the operator* p_h *satisfying Eqs.* (1-7) *is equal to*:

$$p_h = J^{-1} A' r_h^{0\prime} (r_h{}^0 A J^{-1} A' r_h^{0\prime})^{-1} A_h \tag{1-9}$$

Proof. The only point to verify is that $r_h{}^0 A J^{-1} A' r_h^{0\prime}$ is an isomorphism from F_h' onto F_h. (Then, it is clear that $u = p_h u_h$ is a solution of Eq. (1-3)). But J^{-1} is the canonical isometry from V' onto V and it satisfies $((f, g))_{V'} = (J^{-1} f, g)$ (where $((f, g)_{V'}$ denotes the scalar product in the Hilbert space V' and where $\| f \|_{V'} = ((f, f))_{V'}^{1/2}$ is the dual norm to $\| u \|$).

Then, if we supply F_h' with the norm $\| f_h \|_{F_h'} = \| A' r_h^{0\prime} f_h \|_{V'}$, we see that $r_h{}^0 A J^{-1} A' r_h^{0\prime}$ is the canonical isometry from F_h' onto F_h. Thus it is invertible.

Remark 1.2. The above results can be extended to the case where $\| u \|$ is no longer defined by a nonnegative Hermitian form. It is enough to assume that $\| u \|^2$ is Gâteaux-differentiable and to replace J (defined by Eq. (1-1)) by the differential defined by:

$$(Ju, v) = \lim_{\lambda \to 0} \lambda^{-1} (\| u \|^2 - \| u - \lambda v \|^2). \tag{1-10}$$

Then Theorem 1-1 holds. Corollary 1.3 holds if we assume that V and V' are uniformly convex for the norm $\| u \|$.

Remark 1.3. Theorem 1.1 implies that the subspace of approximants $P_h = \bigcup_{u_h \in F_h} p_h u_h$ is contained in the subspace $\hat{P}_h$ defined by

$$J \hat{P}_h = A' r_h^{0\prime} F_h'. \tag{1-11}$$

2. A Commutation Formula (I)

We consider here the particular case where the seminorm $\| u \|$ is $| Au |$ (where $| f |$ denotes the norm in the Hilbert space F). Let us denote by K the canonical isometry from F onto F' and by p_h^0 the operator from F_h into F defined by

$$p_h^0 u_h = K^{-1} r_h^{0\prime} (r_h^0 K^{-1} r_h^{0\prime})^{-1} u_h , \qquad \text{where } r_h^0 \text{ maps } F \text{ onto } F_h . \quad (2\text{-}1)$$

By Corollary 1.3, $p_h^0 u_h$ is the unique solution of the problem of optimal interpolation in F:

$$r_h^0 p_h^0 u_h = u_h \quad \text{and} \quad | p_h^0 u_h | \leqslant | v | \quad \text{for every } v \text{ such that} \quad r_h^0 v = u_h . \quad (2\text{-}2)$$

Moreover, $p_h^0 r_h^0$ is the orthogonal projector whose kernel is $\mathrm{Ker}(r_h^0)$. We shall express in this section the subset $p_h u_h$ in terms of the operator p_h^0.

Theorem 2.1. *Let us assume that the range of $r_h^0 A$ is closed. Let $\| u \| = | Au |$ (where $| \ |$ is the norm in the Hilbert space F) and let p_h^0 be defined by* (2.1). *Then the subset $p_h u_h$ satisfies*

$$A p_h u_h = p_h^0 A_h u_h + (1 - p_h^0 r_h^0)\, G(A)^{\oplus}, \quad (2\text{-}3)$$

where $G(A)^{\oplus}$ is the (Hilbert space) orthogonal complement of the range $G(A)$ of A.

Proof. In this case, the operator J is equal to $A'KA$. By Theorem 1.1, u belongs to $p_h u_h$ if and only if $A'KAu = A' r_h^{0\prime} f_h$. In other words, we can write this equation in the following form:

$$Au = K^{-1} r_h^{0\prime} f_h + K^{-1} z \qquad \text{where} \qquad z \in \ker(A'). \quad (2\text{-}4)$$

Applying now r_h^0 to both sides of this relation and using Eqs. (1-2)(i) and (2-1), we deduce that

$$Au = p_h^0 A_h u + (1 - p_h^0 r_h^0) z \quad \text{where } z \text{ belongs to } \ker(A'). \quad (2\text{-}5)$$

Conversely, if u is a solution of Eq. (2-5), we find that $r_h^0 A u = A_h u_h$ and that $A'KAu = Ju$ belongs to the range of $A' r_h^{0\prime}$. Therefore, u belongs to $p_h u_h$. It remains to prove that $K^{-1} z$ belongs to $G(A)^{\oplus}$. But $\ker(A')$ is the annihilator of the range $G(A)$ of A, and the canonical isometry K is an isomorphism from the (Hilbert space) orthogonal complement $G(A)^{\oplus}$ of $G(A)$ onto its annihilator $G(A)^{\perp}$ in F'.

COROLLARY 2.1. *Let us assume that the range of $r_h{}^0 A$ is closed and that*

$$G(A)^{\oplus} \subset P_h{}^0, \quad (\textit{where } G(A) \textit{ and } P_h{}^0 \textit{ are the ranges of } A \textit{ and } p_h{}^0, \textit{ respectively.}) \tag{2-6}$$

Then

$$A p_h u_h = p_h{}^0 A_h u_h\,. \tag{2-7}$$

Furthermore, if

$$\ker A = 0 \qquad \textit{and} \qquad p_h{}^0 G(A_h) \subset G(A) \tag{2-8}$$

(*where $G(A_h)$ is the range of A_h*), *then there exists a unique solution u belonging to $p_h u_h$ and the operator p_h satisfies the commutation formula* (2-7).

(The proof is obvious.)

In particular, $G(A)^{\oplus} = 0$ if the range of A is dense in F.

COROLLARY 2.2. *Let us assume that the range of $r_h{}^0 A$ is closed and that*

$$\begin{aligned} &\text{(i)} && \ker(A) = 0, \\ &\text{(ii)} && p_h{}^0 G(A_h) \subset G(A) \quad \text{and} \quad r_h{}^0 G(A) \subset G(A_h). \end{aligned} \tag{2-9}$$

Then there exists a linear operator r_h from V onto V_h such that

$$\begin{aligned} &\text{(i)} && p_h r_h \textit{ is the orthogonal projector (in } V\textit{) onto } P_h = p_h V_h\,, \\ &\text{(ii)} && A p_h r_h u = p_h{}^0 r_h{}^0 A u. \end{aligned} \tag{2-10}$$

Proof. The best approximant $p_h r_h u \in P_h$ of u (in V) satisfies

$$(Au - A p_h r_h u, A p_h v_h)_F = 0, \qquad \text{for every} \quad v_h \in V_h\,. \tag{2-11}$$

Using the commutation formula (2-7), we deduce that

$$A_h{}' p_h^{0\prime} K(Au - p_h{}^0 r_h u) = 0. \tag{2-12}$$

We can write this equality in the form

$$(p_h^{0\prime} K p_h{}^0)\, A_h r_h u = p_h^{0\prime} K A u + z_h\,, \qquad \text{where } z_h \text{ belongs to } \ker(A_h{}'). \tag{2-13}$$

By Eq. (2-1), we notice that we can write $r_h{}^0 = (p_h^{0\prime} K p_h{}^0)^{-1} p_h{}^0 K$. Thus, we have obtained the relation

$$A_h r_h u - r_h{}^0 A u = (p_h^{0\prime} K p_h{}^0)^{-1}\, z_h\,, \qquad \text{where} \quad z_h \in \ker(A_h{}'). \tag{2-14}$$

Since $r_h{}^0 G(A) \subset G(A_h)$, $A_h r_h u - r_h{}^0 A u$ belongs to the range $G(A_h)$ of A_h. On the other hand, $p_h^{0\prime} K p_h{}^0$ is the canonical isometry from F_h onto $F_h{}'$ (when either F_h is supplied with the norm $| p_h{}^0 f_h |$ or $F_h{}'$ is supplied with the norm $| r_h^{0\prime} f_h |_{F'}$). Therefore, when z_h ranges over $\ker(A_h{}') = G(A_h)^\perp$, $(p_h^{0\prime} K p_h{}^0)^{-1} z_h$ ranges over the (Hilbert space) orthogonal complement $G(A_h)^\oplus$ of $G(A_h)$ in F_h.

So, $r_h{}^0 A u = A_h r_h u$ since

$$A_h r_h u - r_h{}^0 A u \in G(A_h) \cap (A_h)^\oplus = 0. \tag{2-15}$$

Therefore, $A p_h r_h u = p_h{}^0 A_h r_h u = p_h{}^0 r_h{}^0 A u$.

Remark 2.2. The following result has important consequences: If $f - p_h{}^0 r_h{}^0 f$ converges to 0 in F as $h \to 0$, then $u - p_h r_h u$ converges to 0 in V (supplied with the seminorm $\| u \| = | Au |$).

We notice also that the convergence properties do not depend on the choice of a particular operator A_h (satisfying (2-9)(ii)). In particular, the error functions of p_h *do not depend on the choice of* A_h; if U is a subspace of V supplied with a stronger topology, the error function $e_U{}^V(p_h)$ is defined by

$$\begin{aligned} e_U{}^V(p_h) &= \| 1 - p_h r_h \|_{L(U,V)} = \sup_{u \in U} \| u - p_h r_h u \|_V / \| u \| \\ &= \sup_{u \in U} |(1 - p_h{}^0 r_h{}^0) A u | / \| u \|_U . \end{aligned} \tag{2-16}$$

Let us suppose now that V is contained in F with a stronger topology and that both A and A_h are isomorphisms. We thus can define the error function $e_V{}^F(p_h)$:

$$e_V{}^F(p_h) = \sup_{u \in U} \inf_{v_h \in V_h} | u - p_h v_h | / \| u \|_V \tag{2-17}$$

and, if V_h is a finite-dimensional space, the stability function $s_V{}^F(p_h)$:

$$s_V{}^F(p_h) = \sup_{v_h \in V_h} \| p_h u_h \|_V / | p_h v_h | = \sup_{v_h \in V_h} | p_h{}^0 A_h r_h | / | p_h v_h |. \tag{2-18}$$

We have characterized these functions as eigenvalues of operators (cf. Ref. [3]). We deduce from Theorem 3.3 of Ref. [3] and from the formula $p_h = A^{-1} p_h{}^0 A_h$ the following corollary

COROLLARY 2.3. *Let us assume that V is contained in F with a stronger topology and that the operators A and A_h are isomorphisms.*

Then the error function $e_V{}^F(p_h)$ and the stability function $s_V{}^F(p_h)$ depend only on $r_h{}^0$ and A and are independent of the choice of the operator A_h.

Proof. Since $s_V{}^F(p_h)^2 = \sup_{v_h} \| p_h v_h \|_V^2 / | p_h v_h |^2$ is achieved at a point u_h of the unit-ball of V_h supplied with the norm $| p_h v_h |$, the functional $| p_h{}^0 A_h v_h | / | p_h v_h |$ is differentiable at u_h and its derivative vanishes (cf. Ref. [3, Section 3]). We thus deduce that

$$A_h' p_h^{0\prime} K p_h{}^0 A_h u_h = s_V{}^F(p_h)^2 (p_h' K p_h) u_h = s_V{}^F(p_h)^2 (A_h' p_h^{0\prime} A'^{-1} K A^{-1} p_h{}^0 A_h) u_h .$$

Since A_h is an isomorphism, this amounts to saying that $s_V{}^F(p_h)^2$ is the largest eigenvalue of the operator $(p_h^{0\prime} A'^{-1} K A^{-1} p_h{}^0)^{-1} (p_h^{0\prime} K p_h{}^0)$ and does not depend on A_h.

On the other hand, the error function $e_V{}^F(p_h)$ is equal to

$$\sup_{u \in V} | u - p_h s_h u | / \| u \|,$$

where $p_h s_h$ is the orthogonal projector from F onto $P_h = p_h V_h$. It satisfies $p_h s_h = p_h (p_h' K p_h)^{-1} p_h' K u$. Since $p_h = A^{-1} p_h{}^0 A_h$, we see that $p_h s_h$ equals $A^{-1} p_h{}^0 (p_h^{0\prime} A'^{-1} K A^{-1} p_h{}^0)^{-1} p_h^{0\prime} A'^{-1} K$ and does not depend on A_h.

3. Examples

3.1. *Construction of approximants of Sobolev spaces $H^m(R)$.*

We consider the situation where $F = L^2(R)$ and V is the Sobolev space $H^m(R)$ of functions $u \in L^2(R)$ such that the (weak) derivative $D^m u \in L^2(R)$.

We choose for A the operator D^m. This operator is one-to-one and its range $G(D^m)$ is dense in F.

The discrete analogues we shall choose are the spaces $V_h = F_h = l^2(Z)$ of square sommable sequences $u_h = (u_h{}^j)_{j \in Z}$ defined on the ring Z of integers, and the operator $A_h = \nabla_h{}^m$ of finite differences:

$$(A_h u_h)^j = (\nabla_h{}^m u_h)^j = \sum_{R=0}^{m} (-1)^k \binom{m}{k} u_h^{j-k}. \tag{3-1}$$

We introduce the operator $r_h{}^0$ defined by

$$(r_h{}^0 u)^j = h^{-1} \int_{jh}^{(j+1)h} u(x)\, dx. \tag{3-2}$$

The assumptions of Corollary 2.2 are satisfied: the operator $p_h{}^m$ satisfying

$$\begin{aligned} &\text{(i)} \quad r_h{}^0 D^m p_h{}^m u_h = \nabla_h{}^m u_h , \\ &\text{(ii)} \quad | D^m p_h{}^m u_h | \leqslant | D^m v | \text{ for every } v \text{ such that } r_h{}^0 D^m v = \nabla_h{}^m u_h , \end{aligned} \tag{3-2}$$

is the operator satisfying the following commutation formula:

$$D^m p_h^m u_h = p_h^0 \nabla_h^m u_h , \tag{3-3}$$

where p_h^0 is defined by Eq. (2-1). A simple computation shows that

$$p_h^0 u_h = \sum u_h^j e_{jh} \text{ where } e_{jh}(x) \text{ is the characteristic function of the interval } (jh, (j+1)h). \tag{3-4}$$

But the solution of Eq. (3-3) is well known (cf. [4, 5, 7, 8, 9]).

Since p_h^m is an operator of the form $p_h^m u_h = \sum u_h^j \pi_{mh}^j(x)$ and since $p_h^0 u_h = \sum_j (\sum_k a_k^j \theta_{kh}) u_h^j$ (where $a_k^j = h^{-m}(-1)^{k-j}\binom{m}{k-j}$), we have to solve the differential equation

$$D^m(\pi_{mh}^j(x)) = \sum_k a_k^j e_{kh}(x) = \nabla_h^m e_{jh}(x). \tag{3-5}$$

The solution of Eq. (3-5) is $\pi_{mh}^j(x) = \pi_m((x/h) - j)$ where $\pi_m(x)$ is the $(m+1)$-th fold convolution of the characteristic function of the interval (0,1). The support of this function is contained in the interval $(0, m+1)$ and its restriction to each interval $(k, k+1)$ is a polynomial of degree m.

Therefore, here again, "spline-functions" (i.e., piecewise-polynomial functions) are the optimal solutions of a problem of approximation in the Sobolev spaces $H^m(R)$.

Remark 3.1. It is possible to replace ∇_h^m by any other operator A_h. In this case, the operator p_h satisfying Eqs. (3-2) or (3-4) maps also a sequence into the space of piecewise-polynomials. By Corollary 2.2, the convergence properties are the same. But the support of the function $\pi_{mh}^j(x)$ will be no longer compact.

The fact that $\pi_m(x)$ has a compact support plays an important role. (When these approximations are used in differential problems, the size of the support of $\pi_{mh}^j(x)$ is related to the number of nonzero diagonals of the approximated matrix).

Remark 3.2. We obtain analogous results by replacing the regular intervals $(jh, (j+1)h)$ by irregular intervals and the operator ∇_h^m by the divided-difference operator (cf. [7–9].)

Remark 3.3. It is possible to replace D^m by any other nondegenerate differential operator of order m which is one-to-one and which has a dense range. Among the operators A_h, we would have to choose the one which minimizes the size of the support of the functions $\pi_h^j(x)$ such that $p_h u_h = \sum u_h^j \pi_h^j(x)$. (If we choose $A_h = 1$, we have to solve a problem of optimal interpolation; it is already known that, in this case, the functions π_h^j do not have a compact support for $m > 1$) cf. e.g., [1, 8]).

3.2. *Construction of approximants in the domain of a degenerate operator.* We shall give an example where A is a degenerate differential operator: We choose

$$F = L^2(-1, +1),$$
$$V = \{u \in L^2(-1, +1) \text{ such that } (1 - x^2)\, Du \in L^2(-1, +1)\} \tag{3-6}$$

and we take A defined by $Au = (1 - x^2)\, Du$.

The discrete analogs will be

$$V_h = F_h = l^2(Z), \qquad A_h = \nabla_h . \tag{3-7}$$

We define $r_h{}^0$ by

$$(r_h{}^0 u)^j = (\tanh((j+1)h) - \tanh(jh))^{-1} \int_{\tanh(jh)}^{\tanh((j+1)h)} u(x)\, dx. \tag{3-8}$$

The assumptions of Corollary 2.2 are satisfied: the operator p_h which satisfies

$$\text{(i)} \quad r_h{}^0(1 - x^2)\, Dp_h u_h = \nabla_h u_h ,$$
$$\text{(ii)} \quad |(1 - x^2)\, Dp_h u_h| \leqslant |(1 - x^2)\, Dv| \quad \text{for every } v \text{ such that } r_h{}^0 Dv = \nabla_h u_h , \tag{3-8}$$

is the one satisfying

$$(1 - x^2)\, Dp_h u_h = p_h{}^0 \nabla_h u_h \tag{3-9}$$

where $p_h{}^0$ (defined by Eq. (2-1)) satisfies

$$p_h{}^0 u_h = \sum u_h{}^j e_{jh}(x) \tag{3-10}$$

where θ_{jh} is the characteristic function of the interval $(\tanh(jh), \tanh((j+1)h))$. If we write $p_h u_h$ in the form $\sum u_h{}^j \pi_{jh}(x)$, the function π_{jh} is the solution of the differential equation

$$(1 - x^2)\, D\pi_{jh}(x) = h^{-1}(\theta_{jh}(x) - \theta_{(j+1)h}(x)). \tag{3-11}$$

Thus,

$$\pi_{jh}(x) = h^{-1}(\operatorname{arctanh}(x - j))\, \theta_{jh} - h^{-1}(\operatorname{arctanh}(x - j - 2))\, \theta_{(j+1)h} . \tag{3-16}$$

Therefore, the space P_h of linear combinations of the functions π_{jh} is a space of approximants of the domain $(1 - x^2)D$ which are optimal in the sense of Eq. (3-8).

If we use these approximants to approximate differential equations of the form $-(1 - x^2) D(a(x) Du) + 2xb(x) Du = f$ where $a(x)$ and $b(x) \geqslant c > 0$, we shall obtain finite-differences schemes whose matrices have 3 nonzero diagonals (cf., e.g., [4–5], for the construction of these schemes).

This method can be applied to many other situations.

3.3. *Construction of approximants in spaces of functions of several variables.*

We take $F = L^2(R^n)$. Let V be the space of the functions u of $L^2(R^n)$ such that $D^q u = D_1^{q_1} \cdots D_n^{q_n} u$ belongs to $L^2(R^n)$. We consider the following operator:

$$Au = D^q u = D_1^{q_1} \cdots D_n^{q_n} u. \tag{3-17}$$

The discrete analogs will be $V_h = F_h = l^2(Z^n)$ and the operator A_h defined by

$$A_h u_h = \nabla_{h_1}^{q_1} \cdots \nabla_{h_n}^{q_n} u_h = \nabla_h^{\,q} u_h\, . \tag{3-18}$$

We define r_h^0 by $(r_h^0 u)^j = (h_1 \cdots h_n)^{-1} \int_{m_{jh}} u(x)\, dx$ where

$$m_{jh} = \prod (j_k h_k\, , (j_k + 1)\, h_k).$$

Therefore the approximants $p_h u_h$ which satisfy

$$\begin{aligned} &\text{(i)} \quad r_h^0 D^q p_h u_h = \nabla_h^{\,q} u_h\, , \\ &\text{(ii)} \quad | D^q p_h u_h | \leqslant | D^q v | \text{ for every } v \text{ such that } r_h^0 D^q v = \nabla_h^{\,q} v, \end{aligned} \tag{3-19}$$

are the approximants defined by $p_h u_h = \sum u_h^{\,j} \pi_q(x/h - j)$ with $\pi_q(x) = \pi_{q_1(x_1)} \cdots \pi_{q_n}(x_n)$ (where $\pi_q(t)$ is the $(q + 1)$-th fold convolution of the characteristic function of the interval $(0, 1)$). These approximants are studied and used, for instance, in Refs. [4, 5].

Let us consider the case where

$$\begin{aligned} A &= D^q(D_1 + \cdots + D_n), \qquad A_h = \nabla_h^{\,q} \hat{\nabla}_h\, ; \\ (\hat{\nabla}_h u_h)^j &= h^{-1}(u_h^{\,j} - u_h^{j_1 - 1, \ldots, j_n - 1}). \end{aligned} \tag{3-20}$$

The spaces $V, F, V_h\, , F_h$ and the operator r_h^0 are the same. Then the approximants $p_h u_h$ which satisfy

$$\begin{aligned} &\text{(i)} \quad r_h^0 D^q(D_1 + \cdots + D_n)\, p_h u_h = \nabla_h^{\,q} \hat{\nabla}_h u_h\, , \\ &\text{(ii)} \quad | D^q(D_1 + \cdots + D_n)\, p_h u_h | \leqslant | D^q(D_1 + \cdots + D_n) v | \\ &\qquad \text{for every } v \text{ such that } D^q(D_1 + \cdots + D_n) v = \nabla_h^{\,q} \hat{\nabla}_h u_h\, , \end{aligned} \tag{3-21}$$

are the approximants defined by $p_h u_h = \sum u_h^j \mu_q^j((x/h) - j)$ where

$$\mu_q^j(x) = \int_0^1 \pi_q(x - t)\, dt. \tag{3-22}$$

(These approximants were introduced in Ref. [6].)

In these two cases, the optimal approximants are piecewise-polynomials of multi-degrees q and $q + 1$, respectively.

Finally, let us consider the case where

$$V = H^{2m}(R^n), \qquad F = L^2(R^n) \qquad \text{and} \quad A = (-\Delta/4\pi^2 + 1)^m. \tag{3-23}$$

We introduce the following analogs:

$$V_h = F_h = l^2(Z^n); \qquad A_h = (a_k^j) \text{ is an infinite matrix.} \tag{3-24}$$

We consider r_h^0 defined by $(r_h^0 u)^j = (h_1 \cdots h_n)^{-1} \int_{m_{jh}} u(x)\, dx$; then the operator p_h^0 defined by Eq. (2-1) satisfies $p_h^0 u^h = \sum u_h^j \theta_{jh}$, where θ_{jh} is the characteristic function of m_{jh}.

Then Corollary 2.2 implies that the approximants $p_h u_h$ satisfying

$$\begin{aligned} &\text{(i)} \quad r_h^0 A p_h u_h = A_h u_h \\ &\text{(ii)} \quad | A p_h u_h | \leqslant | A v | \text{ for every } v \text{ such that } r_h^0 A v = A_h u_h \end{aligned} \tag{3-25}$$

satisfy also

$$p_h u_h = \sum_j u_h^j \pi_{jh} \tag{3-26}$$

where $\pi_{jh} = \sum_k a_k^j A^{-1} \theta_{kh}(x)$.

Let $\mu_m(x) = (2\pi^m/m!)\, Pf(| x |^{m-(n/2)} K_{m-(n/2)}(2\pi | x |)$ (where K is a Bessel function, cf. [10, p. 47]) be the fundamental solution of A. Then the approximants π_{jh} are

$$\pi_{jh}(x) = \sum_k a_k^j (\mu_m * \theta_{kh})(x) \tag{3-27}$$

where * denotes convolution.

4. Commutation formula (II)

In this section, we assume that both $A \in L(V, F)$ and $r_h^0 \in L(F, F_h)$ are onto. We supply V with the norm $\| u \|$ defined by the scalar product

$$((u, v)) = (Ju, v) \tag{4-1}$$

where J is the canonical isometry from V onto V'. Then $K = (AJ^{-1}A')^{-1}$ is the canonical isometry from F onto F' when F' is supplied with the norm $\| A'f \|_{F'}$.

Therefore, since r_h^0 maps F onto F_h, we can associate with it the operator p_h^0 defined by

$$p_h^0 = K^{-1} r_h^{0\prime} (r_h^0 K^{-1} r_h^{0\prime})^{-1} \tag{4-2}$$

which satisfies

$$\begin{aligned} &\text{(i)} \quad r_h^0 p_h^0 u_h = u_h \quad &&\text{for every } u_h \in F_h\,, \\ &\text{(ii)} \quad \| p_h^0 u_h \|_F \leqslant \| v \|_F \quad &&\text{for every } v \text{ such that } r_h^0 v = u_h\,. \end{aligned} \tag{4-3}$$

By Corollary 1.3, the solution $p_h u_h$ of the problem

$$\begin{aligned} &\text{(i)} \quad r_h^0 A p_h u_h = A_h u_h\,, \\ &\text{(ii)} \quad \| p_h u_h \| \leqslant \| v \| \text{ for every } v \text{ such that } r_h^0 A v = A_h u_h\,, \end{aligned} \tag{4-4}$$

is given by $p_h = J^{-1} A' K p_h^0 A_h$. We shall give another interpretation of this formula. Let us introduce the operator L defined by:

$$(Lu, v) = ((u, v)) \quad \text{for every } u \in V, \quad \text{and every } v \in Z = \ker A. \tag{4-5}$$

The operator L maps V into the dual Z' of Z.

THEOREM 4.1. *Let us assume that A and r_h^0 are surjective and that $((u, v))$ is a nondegenerate scalar product of the Hilbert space V. Then the solution $p_h u_h$ of* (4-4) *is the solution of*

$$\begin{aligned} &\text{(i)} \quad L p_h u_h = 0, \\ &\text{(ii)} \quad A p_h u_h = p_h^0 A_h u_h\,, \end{aligned} \tag{4-6}$$

where the operators p_h^0 and L are defined by Eqs. (4-2) *and* (4-5).

Proof. Since $J p_h u_h = A' K p_h^0 u_h$, we deduce that

$$(J p_h u_h, v) = (L p_h u_h, v) = (K p_h^0 A_h u_h, Av) = 0 \quad \text{for every} \quad v \in Z = \ker A. \tag{4-7}$$

So $L p_h u_h = 0$. On the other hand,

$$A p_h u_h = (AJ^{-1}A')\, K p_h^0 A_h u_h = K^{-1} K p_h^0 A_h u_h = p_h^0 A_h u_h\,. \tag{4-8}$$

Therefore, $p_h u_h$ is the solution of the problem (4-6). Conversely, let us assume that $p_h u_h$ satisfies the problem (4.6). Since $L p_h u_h = 0$, $J p_h u_h$ belongs to the annihilator $Z^\perp$ of Z (in V'). Since Z is the kernel of A and since A is onto, A' is an isomorphism from F' onto $Z^\perp$. Thus, there exists a unique element Bp_hu_h of $Z^\perp$ such that

$$Jp_hu_h = A'Bp_hu_h \,. \tag{4-9}$$

But KAJ^{-1} is a left inverse of A' since $KAJ^{-1}A' = KK^{-1} = 1$. Applying this operator to both members of Eq. (4-9), we deduce that

$$Bp_hu_h = KAp_hu_h = Kp_h{}^0A_hu_h \,. \tag{4-10}$$

Thus, $Jp_hu_h = A'Bp_hu_h = A'Kp_h{}^0A_hu_h$. By Corollary 1.3, this implies that p_hu_h is the solution of the problem (4-4).

Remark 4.1. Since $Z' = V'/Z^\perp$, Equation (4-6)(i) amounts to saying that

$$Lp_hu_h \in Z^\perp. \tag{4-11}$$

COROLLARY 4.1. *Let us assume the hypotheses of Theorem* 4.1. *Let* r_h *be the operator from* V *onto* V_h *such that* p_hr_h *is the orthogonal projector from* V *onto* $P_h = p_hV_h$. *Then, if* A_h *is onto, the following commutation formula holds*:

$$A_hr_hu = r_h{}^0A_hu_h \,. \tag{4-12}$$

Therefore, the projectors p_hr_h *and* $p_h{}^0r_h{}^0$ *are related by*

$$\begin{aligned} &\text{(i)} && Lp_hr_hu = 0, \\ &\text{(ii)} && Ap_hr_hu = p_h{}^0r_h{}^0Au. \end{aligned} \tag{4-13}$$

Proof. Since $(Ju - Jp_hr_hu, p_hv_h) = 0$ for any $v_h \in V_h$, we deduce that $p_h'Ju = p_h'Jp_hr_hu$. Since $Jp_h = A'Kp_h{}^0A_h$, this equation can be written in the form

$$A_h'(p_h^{0\prime}KAu - p_h^{0\prime}Kp_h{}^0A_hr_hu). \tag{4-14}$$

This implies that

$$A_hr_hu = (p_h^{0\prime}Kp_h{}^0)^{-1}\, p_h^{0\prime}KAu, \tag{4-15}$$

since we have assumed that A_h is onto. Now, it is easy to check that we can write $r_h{}^0 = (p_h^{0\prime}Kp_h{}^0)^{-1}p_h^{0\prime}K$. Thus, Eq. (4-12) holds.

EXAMPLE 4.1. Assume that $V = D(A)$ is the domain of a closed unbounded operator A of a Hilbert space F (identified with its dual). We assume, moreover, that

$$\begin{aligned} &A \text{ maps } V = D(A) \text{ onto } F; \\ &D(A) \text{ is supplied with the graph norm } \| u \|^2 = (| Au |_F^2 + | u |_F^2)^{1/2}. \end{aligned} \tag{4-16}$$

Let Z be the kernel of A. Then the operator L is defined by

$$(Lu, v) = ((u, v)) = (Au, Av) + (u, v) = (u, v) \qquad \text{for every} \quad v \in Z. \tag{4-17}$$

We thus deduce the following corollary:

COROLLARY 4.2. *Let V be the domain of a closed unbounded operator A of a Hilbert space F. Let us assume the conditions* (4-16). *Then the solution $p_h u_h$ of the problem* (4-4) *is the unique solution of*

$$\begin{aligned} &\text{(i)} && p_h u_h \in Z^{\perp} \quad (\textit{where } Z = \ker A), \\ &\text{(ii)} && A p_h u_h = p_h{}^0 A_h u_h . \end{aligned} \tag{4-18}$$

Proof. By Eq. (4-6)(i), $Lp_h u_h = 0$. By Eq. (4-17), this amounts to

$$(Lp_h u_h , v) = (p_h u_h , v) = 0 \qquad \text{for every} \quad v \in Z. \tag{4-19}$$

This implies (4-18)(i)

Let us consider a more general situation. We assume that there exist two continuous operators A and δ such that:

$$\begin{aligned} &\text{(i)} && A \text{ maps } V \text{ onto a Hilbert space } F, \\ &\text{(ii)} && \delta \text{ maps } V \text{ onto a Hilbert space } T, \\ &\text{(iii)} && V \text{ is the direct sum of } V_0 = \ker \delta \text{ and } Z = \ker A. \end{aligned} \tag{4-20}$$

We supply the space V with the scalar product

$$((u, v)) = (Au, Av) + \langle \delta u, \delta v \rangle, \tag{4-21}$$

where $(\ ,\ )$ and $\langle\ ,\ \rangle$ are the scalar products of F and T, respectively. Then V is a Hilbert space for this new scalar product.

COROLLARY 4.3. *Let us assume conditions* (4-20) *and Eq.* (4-21). *If $r_h{}^0$ maps F onto F_h, the solution $p_h u_h$ of the problem* (4-4) *is the solution of*

$$\begin{aligned} &\text{(i)} && \delta p_h u_h = 0, \\ &\text{(ii)} && A p_h u_h = p_h{}^0 A_h u_h . \end{aligned} \tag{4-22}$$

Proof. In this case, the operator L is defined by

$$(Lu, v) = \langle \delta u, \delta v \rangle = (\delta' \delta u, v) \qquad \text{for any} \quad v \in Z. \tag{4-23}$$

So the equation $Lp_h u_h = 0$ is equivalent to $\delta' \delta p_h u_h \in Z^{\perp}$.

On the other hand, δ' is an isomorphism from T' onto $V_0{}^{\perp}$. Therefore, $\delta' \delta p_h u_h \in Z^{\perp} \cap V_0{}^{\perp} = 0$ (since V is the direct sum of V_0 and Z). This implies that $\delta p_h u_h = 0$.

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